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# Conformal foliations and constraint quantization

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**Abstract**

We show that the ***Classical Constraint Algebra*** of a ***Parametrized Relativistic Gauge System*** induces a natural structure of ***Conformal Foliation on*** a ***Transversal Gauge.*** Using the theory of conformal foliations, we provide a natural ***Factor Ordering*** for the quantum operators associated to the canonical quantization of such gauge system.

***Keywords:*** Parametrized relativistic gauge system; Conformal foliations; Canonical quantization
1991 ***MSC:*** 58F06, 81S10, 81T70, 81T13

## 1. Introduction

The purpose of this paper is to show that the theory of *Conformal Foliations,* as developed a few years ago by Montesinos, Vaisman, and others, helps the understanding of *Constraint (Dirac) Quantization* of *Parametrized Relativistic Gauge Systems* by providing a natural solution for the so-called *Factor Ordering Problem* in the *Quantum Operators* corresponding to the *Classical Constraints* $H_\alpha = 0 = H$ *(see* discussion in [8,5]).

We follow Hájicẽk/Kuchař's philosophy, developed in a series of papers (see [5–8]), but here we adopt modem differential geometric methods (in the spirit of [12] or [*13*]) to construct the natural quantum operators and deduce the corresponding ***Commutator Algebra.*** We hope that the use of the theory of conformal foliations helps to clarify the essential unique geometric character of Hájicẽk/Kuchař's quantization method, as well as the naturality in the choice of the quantum operators.

The paper is organized as follows. In Section 2, we redefine the model, introduced in [5], of *Parametrized Relativistic Gauge System* (where it will be called only a Gauge ***System).***

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using symplectic geometry (see [1], [3] or [4]). Section 3 discusses the *Transversal Geometry* of the *Gauge Foliation* $\mathcal{F}$, in terms of the choice of a transversal distribution $\mathcal{T}$, and shows that the mathematical structure deduced from the classical *Constraint Algebra* (9), (10), is that of a conformal foliation with *Complementary Form* $\lambda = -\Omega$ (see Definition 4.1). Section 4 reviews the geometry of conformal foliations following closely [10,14]. We introduce a *Conformal Curvature Tensor* $\boldsymbol{C}$ (see (61)) and the unique $g$-Riemannian connection $\nabla$ on $(\mathcal{T}, g)$ such that $\boldsymbol{C}$ is conformally invariant (see Theorem 4.4). In particular, this seems to be the connection introduced in [6] using a substantial different formalism. Then, we define a *Scalar Curvature* $\boldsymbol{S}$ (see Definition 4.11), a *Transversal Laplacian* (see Definition 4.13), constructed from the above-mentioned connection, and deduce some useful identities (see Lemmas 4.8, 4.10 and 4.12). Finally, in Section 5, we implement the *Constraint* (*Dirac*) *Quantization Program*, defining the quantum operators and computing the *Quantum Commutator Algebra* (see Theorems 5.1 and 5.4). The more relevant formulas are deduced in Appendix A.

## 2. Symplectic geometry of gauge systems

The situation we have in mind is the following. Let $M$ be an $(N+1)$-dimensional smooth manifold, $T^*M$ its cotangent bundle, together with its canonical symplectic form $\omega = \mathrm{d}\theta$, and $(C^\infty(T^*M), \{\,,\})$ the Poisson algebra of $C^\infty$-functions on $T^*M$.

Assume that we have a $v$-dim integrable distribution $\mathcal{D}$ (a subbundle of $TM$) and let $\mathcal{F}$ be the corresponding foliation on $M$. The leaves of $\mathcal{F}$ describe *Gauge Equivalent* or physically indistinguishable configurations on $M$. We call the pair $(M, \mathcal{F})$ a *Gauge System.* When $\mathcal{F}$ is simple, then $m = M/\mathcal{F}$ is called the physical space and $T^*m$ the physical phase space of the gauge system.

We consider now a special subalgebra of $(C^\infty(T^*M), \{\,,\})$, consisting of functions, *at most Linear in Momenta*, linearly generated by the following two types of functions.

(i) *Configuration Functions* – functions $F \in C^\infty(M)$, identified with their pullbacks to $T^*M$.

(ii) *Momentum Functions* – to each vector field $X \in \mathcal{X}(M)$ we associate the corresponding *Momentum Function* $J_X$ defined by

$$J_X(\alpha_Q) = \alpha_Q(X_Q), \quad \forall \alpha_Q \in T^*M. \tag{1}$$

The following Poisson brackets characterize the kinematics of the system.

**Proposition 2.1** ([1]).

(i) $\{F_1, F_2\} = 0$,

(ii) $\{F, J_X\} = X \cdot F$,

(iii) $\{J_X, J_Y\} = -J_{[X,Y]}$,

$$\forall F_1, F_2 \in C^\infty(M), \quad \forall X, Y \in \mathcal{X}(M).$$

In particular, we consider momentum functions of the form $J_X$ with $X \in \Gamma\mathcal{D}$ (i.e., $X$ is a vector field on $M$, tangent to $\mathcal{F}$) which we call *Constraint Functions*. They form a subalgebra of $(C^\infty(T^*M), \{\,,\})$ called the (linear) *Gauge Algebra* $\mathcal{G}$ of the system. Using this, we define now the *Constraint Set* $\mathcal{C} \subset T^*M$ by

$$\mathcal{C} = \{\alpha_Q \in T^*M:\ J_X(\alpha_Q) = 0, \forall X \in \Gamma\mathcal{D}\}. \tag{2}$$

Note that $\mathcal{C}$ is the subbundle $\mathcal{D}^0$ of $T^*M$, consisting of the covectors that vanish on $\mathcal{D}$ and, therefore, that $\mathcal{C}$ is a coisotropic submanifold in $T^*M$ (a first class constraint set, in Dirac's terminology). When we consider the closed 2-form $i^*\omega$ (where $i : \mathcal{C} \to T^*M$ is inclusion), we see that it is degenerate and its characteristic distribution $\mathcal{K} = \ker(i^*\omega)$ is integrable and locally generated by the Hamiltonian vector fields associated to the constraint functions. The quotient (if it exists) of $\mathcal{C}$ by the characteristic foliation determined by $\mathcal{K}$, $\mathcal{S} = \mathcal{C}/\mathcal{K}$ has a unique symplectic form $\omega_{\mathcal{S}}$ such that $\pi^*\omega_{\mathcal{S}} = i^*\omega$ (see [1], [3] or [4]):

$$\begin{array}{ccc} \mathcal{C} & \stackrel{i}{\longrightarrow} & T^*M \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathcal{S} = \mathcal{C}/\mathcal{K} & & \end{array}$$

Also if $F_1, F_2 \in C^\infty(T^*M)$ are two functions constant on the leaves of the characteristic foliation, i.e., such that $i^*F_k = \pi^*f_k$ $(k = 1, 2)$, for uniquely determined functions $f_1, f_2 \in C^\infty(\mathcal{S})$, then

$$i^*\{F_1, F_2\} = \pi^*\{f_1, f_2\}_{\mathcal{S}}. \tag{3}$$

Under certain regularity assumptions, we know that $(\mathcal{S}, \omega_{\mathcal{S}})$ is symplectomorphic to $(T^*m, \omega_m = \mathrm{d}\theta_m)$ and so, it is the natural candidate for the physical (or reduced) phase space of the physical system (see [3]).

Now we assume that the intrinsic dynamics of our gauge system is generated by a quadratic function in momenta of the form

$$H = \tfrac{1}{2}G + J_U + V, \tag{4}$$

where $G$ is a contravariant metric (eventually degenerate) on $M$, viewed as a function on $T^*M$, $U \in \mathcal{X}(M)$ is a vector field in $M$ and $V \in C^\infty(M)$. We have the following Poisson brackets, involving the new kind of homogeneous quadratic function $G$.

**Proposition 2.2** ([1]).
(i) $\{G, F\} = -2J_{\mathrm{grad}_G F}$, *where* $\mathrm{grad}_G\, F$ *is the vector field* $G(\mathrm{d}F, .) \in \mathcal{X}(M)$,
(ii) $\{G, J_X\} = L_X G$, *the Lie derivative of* $G$ *in the direction of* $X \in \mathcal{X}(M)$,

$$\forall F \in C^\infty(M), \quad \forall X \in \mathcal{X}(M).$$

Now we consider the equations of motion, which follow from the canonical action principle, for a parametrized gauge system:

$$S(Q^A, P_A; N, N^\alpha) = \int \mathrm{d}t(P_A\dot{Q}^A - N \cdot H - N^\alpha \cdot H_\alpha) \longrightarrow \text{stat.}, \tag{5}$$

where $(Q^A, P_A)$ are the local canonical coordinates in $T^*M$; $N$, $N^\alpha$ Lagrange multipliers and $H_\alpha = J_{X_\alpha}$ are the momentum functions associated to a local basis $X_\alpha$ ($\alpha = 1, \dots, v$) of the *Gauge Distribution* $\mathcal{D}$ on $M$.

Following the physical terminology, we call the $H_\alpha$ the *Supermomentum Functions* and $H$ the *Superhamiltonian* of the (parametrized) gauge system (see [5] for discussion and examples).

Variation with respect to $Q^A$ and $P_A$ yields the Hamiltonian equations; while variation with respect to the *Lapse Function* $N$ and the *Shift Vector Field* $N^\alpha$, leads to the constraints

$$H = 0 = H_\alpha. \tag{6}$$

So, we recover the *Kinematical Constraint Set* $\mathcal{C}$, defined in (2), together with a new *Dynamical Constraint* $H = 0$, that reveals the fact that the system is invariant under external *time* reparametrizations.

Assume that $\mathcal{C}_0 = H^{-1}(0)$ is a submanifold of $T^*M$. Since codim $\mathcal{C}_0 = 1$, $\mathcal{C}_0$ is coisotropic and is foliated by the trajectories of $X_H$ (the Hamiltonian vector field associated to $H$) that represent the evolution of the system. For consistency, we assume that the full set of constraints (6) is preserved by the dynamics, which means that $\mathcal{C}_0 \cap \mathcal{C}$ is coisotropic, i.e., we have

$$\{H, H_\alpha\} = C_\alpha H + C_\alpha^\beta H_\beta, \tag{7}$$

where $H_\alpha = J_{X_\alpha}$ are the supermomenta associated to a local basis $X_\alpha$ of $\mathcal{D}$, $C_\alpha \in C^\infty(M)$ and $C_\alpha^\beta \in C^\infty(T^*(M))$ is at most linear in momenta. If the local basis $X_\alpha$ for $\mathcal{D}$ verifies

$$[X_\alpha, X_\beta] = -\Gamma_{\alpha\beta}^\gamma X_\gamma \tag{8}$$

with $\Gamma_{\alpha\beta}^\gamma \in C^\infty(M)$, then the *Algebra of Constraints* $H = 0 = H_\alpha$ has the following (open) structure:

$$\{H_\alpha, H_\beta\} = \Gamma_{\alpha\beta}^\gamma H_\gamma, \tag{9}$$

$$\{H, H_\alpha\} = C_\alpha H + C_\alpha^\beta H_\beta, \tag{10}$$

$H$ is given by (4) and, working equality (10), using Propositions 2.1 and 2.2, we conclude the following identities for the Lie derivatives in *Gauge Directions*:

$$L_\alpha G = C_\alpha G (\operatorname{mod} I); \tag{11}$$

$$L_\alpha U = C_\alpha U (\operatorname{mod} I); \tag{12}$$

$$L_\alpha V = C_\alpha V, \tag{13}$$

where I is the ideal of the contravariant tensor algebra $\bigotimes TM$, generated by $\Gamma\mathcal{D}$ and $L_\alpha$ is Lie derivative in the direction of $X_\alpha \in \Gamma\mathcal{D}$. So, this Lie derivative rescales the fields by a common factor and adds certain elements from I.

All the conclusions about the physical content of the gauge system described by the above model, must be deduced only from intrinsic data, namely, from the *Kinematical Constraint*

$\mathcal{C}$, determined by the foliation $\mathcal{F}$, and the *Dynamical Constraint* $\mathcal{C}_0$, given by $H = 0$. So, they must be invariant under transformations that preserve these data. We impose also that these transformations preserve the polynomial character (in momenta) of the constraints (this will be needed for the quantization program, as we shall see later, and is related to Van Hove's theorem (see [1, Section 5.4])). So, they must be a combination of the following three types of transformations:

(a) *Change in the local basis for $\mathcal{D}$*:

$$X_\alpha \to \overline{X}_\alpha = \Lambda_\alpha^\beta X_\beta, \tag{14}$$

where $\Lambda_\alpha^\beta \in C^\infty(M)$, $\det \Lambda_\alpha^\beta \neq 0$, with the corresponding change in supermomenta,

$$H_\alpha \to \overline{H}_\alpha = \Lambda_\alpha^\beta H_\beta. \tag{15}$$

(b) *Gauging the superhamiltonian*:

$$H \to \overline{H} = H + \Lambda^\alpha H_\alpha \tag{16}$$

with $\Lambda^\alpha \in C^\infty(T^*(M))$, at most linear in momenta.

(c) *Scaling the superhamiltonian*:

$$H \to \overline{H} = \mathrm{e}^{\Omega} H \tag{17}$$

and $\mathrm{e}^{\Omega} \in C^\infty(M)$. To preserve the signature of the metric we only allow positive scalings.

Note that the above transformations do not leave invariant the superhamiltonian $H$, given by (4). In fact, after an easy computation, we deduce that,.if

$$H \to \overline{H} = \tfrac{1}{2}\overline{G} + J_{\overline{U}} + \overline{V}$$

then

$$\overline{G} = \mathrm{e}^{\Omega}(Q) G \ (\mathrm{mod}\, \boldsymbol{I}), \tag{18}$$

$$\overline{U} = \mathrm{e}^{\Omega}(Q) U \ (\mathrm{mod}\, \boldsymbol{I}), \tag{19}$$

$$\overline{F} = \mathrm{e}^{\Omega}(Q) V. \tag{20}$$

Let us say that two superhamiltonians $H = \frac{1}{2}G + J_U + V$ and $\overline{H} = \frac{1}{2}\overline{G} + J_{\overline{U}} + \overline{V}$, are *Conformally Equivalent* (mod I), if they are related through (18)–(20).

Then we conclude that the above transformations (a)–(c), leave invariant the conformally equivalence class (mod I) of the superhamiltonian.

## 3. Transversal geometry of $\mathcal{F}$

The transversal geometry of $\mathcal{F}$ is infinitesimally modeled by the normal bundle $TM/\mathcal{D}$. We make a choice of a transversal subbundle $\mathcal{T}$ to $\mathcal{D}$,

$$T_Q M = \mathcal{D}_Q \oplus \mathcal{T}_Q \tag{21}$$

which identifies $TM/\mathcal{D} \cong \mathcal{T}$. Vectors on $\mathcal{D}$ are called *Longitudinal* and vectors on $\mathcal{T}$ are called *Transversal*. Associated to (21), we have two projectors

$$\mathrm{Id} = P_{\parallel} \oplus P_{\perp}, \tag{22}$$

a decomposition

$$T_Q{}^*M = \mathcal{D}_Q{}^* \oplus \mathcal{T}_Q{}^* \tag{23}$$

with $\mathcal{T}_Q^* \cong \mathcal{D}_Q^0$, and the two corresponding projectors

$$\mathrm{Id}^* = P_{\parallel}{}^* \oplus P_{\perp}{}^*. \tag{24}$$

We can form various tensor products of these projectors, using them to project the various tensors into longitudinal and transversal parts. In particular, the choice of the *Transversal Gauge* $\mathcal{T}$ allows us to define the *Transversal Superhamiltonian* $H_{\perp}$ by

$$H_{\perp} = \tfrac{1}{2}G_{\perp} + J_{U_{\perp}} + V, \tag{25}$$

where

$$G_{\perp} = (P_{\perp} \otimes P_{\perp})(G) \tag{26}$$

and

$$U_{\perp} = P_{\perp}U \tag{27}$$

are the *Transversal Contravariant Metric* and the *Transversal Vector Potential*, respectively (associated to $G$ and $\mathcal{T}$).

So, the transversal gauge $\mathcal{T}$ fixes a representative of the equivalence class of superhamiltonians, connected by the *Gauging* (b), since the difference between two transversal projections of the same vector belongs to $\mathcal{D}$.

Let us compute the constraint algebra in the transversal gauge $\mathcal{T}$. For this, we compute first $L_\alpha G_{\perp}$ and $L_\alpha U_{\perp}$, using the fact that $L_\alpha$ is a tensor derivation that commutes with contractions,

$$L_\alpha(\mathcal{C}(\mathcal{P}_{\perp} \otimes t)) = \mathcal{C}(L_\alpha \mathcal{P}_{\perp} \otimes t) + \mathcal{C}(\mathcal{P}_{\perp} \otimes L_\alpha t), \tag{28}$$

where $\mathcal{C}$ is a contraction, $\mathcal{P}_{\perp}$ a transversal projector (a tensor product of some $P_{\perp}$ and $P_{\perp}{}^*$) and $t$ some tensor field on $M$.

Define for each $\alpha, \beta \in \{1, \ldots, v\}$ a transversal 1-form (i.e., a form which annihilates longitudinal vectors) $\omega_\alpha^\beta$ by

$$\omega_\alpha^\beta = (L_\alpha P_{\perp}^*)(\theta^\beta), \tag{29}$$

where $\theta^\beta \in \Gamma\mathcal{D}^*$ is the dual basis to $X_\beta \in \Gamma\mathcal{D}$. A computation, using local frame fields for $\mathcal{D}$, $\mathcal{T}$, their duals and also formula (28) (see Appendix A), shows that

$$L_\alpha U_{\perp} = C_\alpha U_{\perp} + \omega_\alpha^\beta(U_{\perp})X_\beta = C_\alpha U_{\perp}\ (\mathrm{mod}\,\boldsymbol{I}), \tag{30}$$

$$L_\alpha G_\perp = C_\alpha G_\perp + V_\alpha^\beta \vee X_\beta = C_\alpha G_\perp \,(\mathrm{mod}\, \boldsymbol{I}), \tag{31}$$

where $V_\alpha^\beta = G(\omega_\alpha^\beta, .)$ is the transversal vector field *metric-equivalent* to $\omega_\alpha^\beta$, and $\vee$ denotes symmetric tensor product.

Finally, using (30) and (31) together with Propositions 2.1 and 2.2, we deduce that

$$\{H_\perp, H_\alpha\} = C_\alpha H_\perp + F_\alpha^\beta H_\beta, \tag{32}$$

where $F_\alpha^\beta \in C^\infty(T^*M)$ is given by

$$F_\alpha^\beta = J_{V_\alpha^\beta} - \omega_\alpha^\beta(U_\perp). \tag{33}$$

Now we arrive at a crucial point – the definition of a 1-form $\Theta$, on $M$, attached to the geometry of the gauge system as specified by the constraint algebra (9) and (10). So, consider the longitudinal 1-form

$$\Theta = C_\alpha \theta^\alpha, \tag{34}$$

where $\theta^\alpha \in \Gamma\mathcal{D}^*$ are dual 1-forms to the $X_\alpha$. It is easy to see that $\Theta$ is a globally well-defined longitudinal 1-form on $M$. We want to compute its *Foliated Derivative* $\mathrm{d}_\mathcal{F}\Theta$ (i.e., the derivative along longitudinal directions, see [16]). For this, we first note that Jacobi Identity

$$\mathrm{cyclsum}\{H, \{H_\alpha, H_\beta\}\} = 0 \tag{35}$$

implies the following identity:

$$\{C_\alpha, H_\beta\} - \{C_\beta, H_\alpha\} = \Gamma_{\alpha\beta}^\gamma C_\gamma. \tag{36}$$

So, recalling that $C_\alpha, \Gamma_{\alpha\beta}^\gamma \in C^\infty(M)$, we deduce that

$$L_\beta C_\alpha - L_\alpha C_\beta = \Gamma_{\alpha\beta}^\gamma C_\gamma \tag{37}$$

and finally

$$\begin{aligned}
\mathrm{d}_\mathcal{F}\Theta(X_\alpha, X_\beta) &= \mathrm{d}\Theta(X_\alpha, X_\beta)\\
&= X_\alpha \cdot \Theta(X_\beta) - X_\beta \cdot \Theta(X_\alpha) - \Theta([X_\alpha, X_\beta])\\
&= L_\alpha C_\beta - L_\beta C_\alpha - \Theta(-\Gamma_{\alpha\beta}^\gamma X_\gamma)\\
&= L_\alpha C_\beta - L_\beta C_\alpha + \Gamma_{\alpha\beta}^\gamma C_\gamma\\
&= 0,
\end{aligned}$$

i.e., $\Theta$ is $\mathrm{d}_\mathcal{F}$-closed. Recall that a kind of *Poincaré Lemma* (see [15]) applies to this case – locally, there exists a function $\Omega$ such that $\Theta = \mathrm{d}_\mathcal{F}\Omega$, i.e.,

$$\Theta(X) = L_X\Omega, \quad \forall X \in \Gamma\mathcal{D}. \tag{38}$$

In particular, for $X = X_\alpha$, we have

$$C_\alpha = L_\alpha \Omega. \tag{39}$$

Consider now the *Transversal Contravariant Metric* $\tilde{g} = G|_{\mathcal{D}^0} = G_\perp|_{\mathcal{D}^0}$ (see Appendix A), that we assume to be nondegenerate. Let $g$ be the associated transversal covariant metric on $\mathcal{T} \cong \mathcal{T}^{**} \cong \mathcal{D}^{0*}$. A computation made in Appendix A (see also the note following Definition 4.1), shows that

$$L_\alpha g = -\Theta(X_\alpha)g. \tag{40}$$

Find a function $\Omega$ that locally verifies (39) and define the rescaled metric $\bar{g}$ by

$$\bar{g} = \mathrm{e}^{\Omega} g \tag{41}$$

Then we have

$$L_\alpha \bar{g} = L_\alpha(\mathrm{e}^{\Omega} g) = (L_\alpha \Omega)\bar{g} + \mathrm{e}^{\Omega} L_\alpha g = C_\alpha \bar{g} - C_\alpha \bar{g} = 0$$

and we see that the rescaled metric is foliated (i.e., constant along the leaves of $\mathcal{F}$), or, put another way, $g$ is locally conformal to a foliated metric.

According to Montesinos (see [10,11] and Section 4) we say that $\mathcal{F}$ is a *Conformal Foliation* and that

$$\lambda = -\Theta \tag{42}$$

is the corresponding *Complementary Form.*

One more point, before closing this section.

Recall that a choice of a transversal subbundle $\mathcal{T}$, fixes a representative $H_\perp$ in the equivalence class of superhamiltonians connected by the *Gauging Transformations* (b). However, we are still free to *rescale* the superhamiltonian according to (c). When we do this, and compute the new structure functions in (10), we see that

$$C_\alpha \to \overline{C_\alpha} = C_\alpha + L_\alpha \Omega \tag{43}$$

and

$$C_\alpha^\beta \to \overline{C_\alpha^\beta} = \mathrm{e}^{\Omega} C_\alpha^\beta. \tag{44}$$

Hence we conclude two things: first, the $\mathrm{d}_{\mathcal{F}}$-cohomology class $[\lambda]$ remains unchanged. In fact

$$\overline{\Theta} - \Theta = (\overline{C_\alpha} - C_\alpha)\theta^\alpha = (L_\alpha \Omega)\theta^\alpha = \mathrm{d}_{\mathcal{F}}\Omega.$$

Secondly, as $\lambda = -\Theta$ is $\mathrm{d}_{\mathcal{F}}$-closed, locally we can choose a function $\Omega$ such that $\mathrm{d}_{\mathcal{F}}\Omega = \lambda$, which implies in particular that $\mathrm{d}_{\mathcal{F}}\Omega(X_\alpha) = L_\alpha \Omega = \lambda(X_\alpha) = -C_\alpha$, and so, by (43), $\overline{C_\alpha} = 0$, for the corresponding rescaled superhamiltonian $\overline{H} = \mathrm{e}^{\Omega} H$.

If we can find globally such a $\Omega$, i.e., if $\lambda$ is $\mathrm{d}_{\mathcal{F}}$-exact, then

$$L_\alpha \overline{G} = 0 = L_\alpha \overline{U} = 0 = L_\alpha \overline{V} \ (\mathrm{mod}\, \boldsymbol{I}) \tag{45}$$

and so the fields $\overline{G}, \overline{U}, \overline{V}$ are projectable in the (leaf) physical space (when it exists). However, note that $\Omega$ is defined up to a function $\omega$ such that $\mathrm{d}_{\mathcal{F}}\omega = 0$, i.e., up to a *basic function* (constant on the leaves of the foliation $\mathcal{F}$).

So, when the foliation is simple, we will have a physical superhamiltonian $h$, defined up to a multiplication by a function $e^w$, with $w \in C^\infty(M)$ a basic function. In other words, the gauge system only determines the conformal geometry in the physical configuration space. Moreover, in this case, if $\pi : M \to m = M/\mathcal{F}$, denotes the canonical projection, then, as $\pi_* = d\pi$ annihilates I, we obtain a class of conformally contravariant tensors on $m$, $(e^w \cdot g, e^w \cdot P_u, e^w \cdot v)$, $w \in C^\infty(m)$ and also a class of conformally physical superhamiltonians

$$\{h\} = \{e^w(g + P_u + v) : w \in C^\infty(m)\}. \tag{46}$$

## 4. Geometry of conformal foliations

In the last section, we have seen that the gauge system determines the structure of conformal foliation on $\mathcal{F}$. To be more specific, we adopt the following definition from [10,11].

**Definition 4.1.** Let $M$ be a manifold, $\mathcal{D}$ an integrable distribution, $\mathcal{T}$ a transversal subbundle to $\mathcal{D}$ and $g$ a covariant metric on $\mathcal{T}$. We say that $(M, \mathcal{D}, \mathcal{T}, g)$ is a *Conformal Foliation*, if there exists a longitudinal 1-form $\lambda$ such that

$$L_X g = \lambda(X) g, \quad \forall X \in \Gamma\mathcal{D}, \tag{47}$$

$\lambda$ is called the *Complementary Form* of $\mathcal{F}$.

**Note**. In Eq. (47), $L_X g$, $X \in \Gamma\mathcal{D}$, is defined as in Eqs. (97), (98) of Appendix A.

Now we collect some facts about conformal foliations (see [10,11,14]).

Let $\nabla$ be a linear connection on the vector bundle $\mathcal{T}$. We define as usual its curvature and torsion, respectively, by

$$\boldsymbol{R}(U, V)Q = ([\nabla_U, \nabla_V] - \nabla_{[U,V]})Q, \tag{48}$$

$$\boldsymbol{T}(U, V) \doteq \nabla_U V_\perp - \nabla_V U_\perp - [U, V]_\perp, \tag{49}$$

where $U, V \in \mathcal{X}(M)$ and $Q \in \Gamma\mathcal{T}$.

Define $\mathcal{D}^{p,q}$ as the space of $(p, q)$-double forms on $M$, i.e., the space of $p$-forms on $M$ with values on transversal $q$-forms (which annihilate vectors on $\mathcal{D}$),

$$\mathcal{D}^{p,q} = \bigwedge^p T^*M \otimes \bigwedge^q \mathcal{T}^*. \tag{50}$$

With R and T we associate two double forms $\boldsymbol{K} \in \mathcal{D}^{2,2}$ and $\boldsymbol{N} \in \mathcal{D}^{2,1}$ defined, respectively, by

$$\boldsymbol{K}(U, V; Q, S) = g(\boldsymbol{R}(U, V)Q, S), \tag{51}$$

$$\boldsymbol{N}(U, V; Q) = g(\boldsymbol{T}(U, V), Q) \tag{52}$$

and also the *Transversal Torsion* $N_\perp$, of $\nabla$ by

$$N_\perp(U,V;.) = N(U_\perp, V_\perp;.) \tag{53}$$

$\forall U, V \in \mathcal{X}(M), \forall Q, S \in \Gamma\mathcal{T}$.

**Definition 4.2.** We say that a linear connection $\nabla$ on the Riemannian vector bundle $(\mathcal{T}, g)$ is $g$-Riemannian, if it verifies the following two conditions:

(i) $$U \cdot g(Q,S) = g(\nabla_U Q, S) + g(Q, \nabla_U S) \tag{54}$$

and

(ii) $$N_\perp = 0 \quad \text{i.e., } \nabla_{U_\perp} V_\perp - \nabla_{V_\perp} U_\perp - [U_\perp, V_\perp]_\perp = 0 \tag{55}$$

$\forall U, V \in \mathcal{X}(M), \forall Q, S \in \Gamma\mathcal{T}$. (Note that we are only assuming $g$ nondegenerate).

For example, if $\tilde{G}$ is a covariant metric on $M$, such that $\tilde{G}|\mathcal{T} = g$, and if $\tilde{\nabla}$ is the Levi-Civitá connection of $\tilde{G}$, then

$$\nabla_U Q = (\tilde{\nabla}_U Q)_\perp \tag{56}$$

is a $g$-Riemannian connection on $(\mathcal{T}, g)$. So $g$-Riemannian connections are not unique and it is difficult to define a natural conformal curvature tensor for the transversal bundle $(\mathcal{T}, g)$.

However, by a careful analysis based on an early work of Kulkarni, Montesinos succeeds in isolating a class of conformally equivalent connections, on which he defines a conformal curvature tensor.

**Note.** For motivation, recall the classical theory: conformal (Weyl) curvature is a tensor field associated to a class of conformally equivalent Riemannian connections and which is conformal invariant,

$$\begin{aligned} g &\longrightarrow \bar{g} = \mathrm{e}^{2f} g, \\ \nabla = \nabla_g &\longrightarrow \overline{\nabla} = \nabla_{\bar{g}}, \text{ defined by} \\ & \qquad \overline{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X \\ & \qquad -g(X,Y)\mathrm{grad}_g f, \\ \boldsymbol{K}(\text{Riemann tensor}) &\longrightarrow \overline{\boldsymbol{K}} = \mathrm{e}^{2f}(\boldsymbol{K} - \ldots) \\ \boldsymbol{C} &= \overline{\boldsymbol{C}} \text{ (Weyl tensor).} \end{aligned}$$

Here the difficulty is in the analog of the second line in the above scheme – what must be $\overline{\nabla} = \nabla \bar{g}$? Montesinos answers the following: let $\nabla$ be any $g$-Riemannian connection on $(\mathcal{T}, g)$. Then there exists a unique $\bar{g}$-Riemannian connection $\overline{\nabla}$, given by

$$\overline{\nabla}_U Q = \nabla_U Q + (Q \cdot \Omega) U_\perp + (U \cdot \Omega) Q - g(U;Q) Z. \tag{57}$$

Its torsion $\overline{N}$ is

$$\overline{N} = \mathrm{e}^{2\Omega}(N - g \wedge (\mathrm{d}\Omega)_\parallel). \tag{58}$$

**Note**. The meaning of the symbols in the above equations is the following: $g$ is interpreted as a (1, 1)-double form defined by $g(U; Q) = g(U_\perp, Q)$; $(\mathrm{d}\Omega)_\parallel \in \mathcal{D}^{1,0}$ is defined by $(\mathrm{d}\Omega)_\parallel(U) = (\mathrm{d}\Omega)(U_\parallel) = (\mathrm{d}_\mathcal{F}\Omega)(U)$; $g \wedge (\mathrm{d}\Omega)_\parallel \in \mathcal{D}^{2,1}$ is defined in the usual way by $g \wedge (\mathrm{d}\Omega)_\parallel(U, V; .) = g(U, .)(\mathrm{d}\Omega)_\parallel(V) - g(V; .)(\mathrm{d}\Omega)_\parallel(U)$ and finally $Z = \mathbf{grad}_\perp \Omega = g^{-1}(\mathrm{d}\Omega|\mathcal{T}, .) \in \Gamma\mathcal{T}$ is the *Transversal Gradient* of $\Omega \in C^\infty(M)$ (see Definition 4.13).

**Definition 4.3.** We call $\overline{\nabla}$, given by (57), the *Connection Conformally Associated* to $\nabla$, by the conformal scaling

$$g \to \bar{g} = \mathrm{e}^{2\Omega} g, \quad \Omega \in C^\infty(M). \tag{59}$$

Based on this class of conformally equivalent connections [10] proceeds in the construction of a conformal curvature tensor, in the following steps:

(1) First, he proves that, given a $w \in \mathcal{D}^{k+1,l+1}$ with $b = v - k - l \geq 1$, there exists only one $\delta w \in \mathcal{D}^{k,l}$ such that (see [10, Theorem 3.3])

$$\boldsymbol{c}(w - g \wedge \delta w) = 0.$$

Here $\boldsymbol{c}$ means contraction: for an $\alpha \in \mathcal{D}^{k+1,l+1}$ we define $\boldsymbol{c}\alpha \in \mathcal{D}^{k,l}$ by

$$\boldsymbol{c}\alpha(U_1, \ldots, U_k; Q_1, \ldots, Q_l) = \sum_a \epsilon_a(E_a, U_1, \ldots, U_k; E_a, Q_1, \ldots, Q_l),$$

where $E_a$ is an orthonormal frame field for $\Gamma\mathcal{T}$ and $\epsilon_a = g(E_a, E_a)$.

With this, he defines a *Conformal Operator* **conf** : $\mathcal{D}^{k+1,l+1} \to \mathcal{D}^{k+1,l+1}$ by

$$\mathbf{conf}\, w = w - g \wedge \delta w \tag{60}$$

so that c($\mathbf{conf}\, w$) =0.

(2) In particular, for $\boldsymbol{K} \in \mathcal{D}^{2,2}$ given by (51), we define the *Conformal Curvature Tensor* C by

$$g(\boldsymbol{C}(U, V)Q, S) = (\mathbf{conf}\boldsymbol{K})(U, V; Q, S). \tag{61}$$

(3) Making a conformal change in $g$: $g \to \bar{g} = \mathrm{e}^{2\Omega} g, \Omega \in C^\infty(M)$, we can prove that

(i) $\quad \mathbf{conf} = \overline{\mathbf{conf}}$

and that

(ii) $\quad \mathbf{conf}\overline{\boldsymbol{K}} = \mathrm{e}^{2\Omega}\mathbf{conf}(\boldsymbol{K} - \boldsymbol{N} \wedge \mathrm{d}\Omega|\mathcal{T})$.

So, denoting by $\overline{\boldsymbol{C}}$ the conformal curvature given by (61) and constructed with $\overline{\nabla}$, given by (57), we see that

$$
\begin{aligned}
e^{2\Omega}(\overline{C}(U,V)Q,S) &= \bar{g}(\overline{C}(U,V)Q,S)\\
&= \overline{\mathbf{conf}\,\overline{K}}(U,V;\,Q,S) \quad \text{by (61)}\\
&= \mathbf{conf}\,\overline{K}(U,V;\,Q,S) \quad \text{by (i) above}\\
&= e^{2\Omega}\mathbf{conf}(K - N\wedge \mathrm{d}\Omega|\mathcal{T})(.,.;.,.) \quad \text{by (ii) above}\\
&= e^{2\Omega}\mathbf{conf}(.,.;.,.) - e^{2\Omega}(N\wedge \mathrm{d}\Omega|\mathcal{T})(.,.;.,.)
\end{aligned}
$$

and we conclude that $C$ is conformally invariant ($C = \overline{C}$) iff

$$\mathbf{conf}(N\wedge \mathrm{d}\Omega|\mathcal{T}) = 0, \quad \forall \Omega \in C^{\infty}(M). \tag{62}$$

Now comes the main point, namely, the existence of a conformally invariant conformal curvature for the transversal bundle $\mathcal{T}$ of $\mathcal{F}$, an exclusive property of conformal foliations (see [10, Theorem 4.2]). For us, the main interest is in the following theorem.

**Theorem 4.4.** *Let* $(M,\mathcal{D},\mathcal{T},g)$ *be a conformal foliation with complementary form* $\lambda$, *and assume that* codim $\mathcal{D} = n+1 \geq 3$. *Then there exists a unique g-Riemannian connection on* $(\mathcal{T},g)$ *such that* $C$ *is conformal invariant.*

*Sketch of proof.* Let $\tilde{G}$ be any metric on $M$ such that $\tilde{G}|\mathcal{T} = g$, and let $\tilde{\nabla}$ be its Levi-Cività connection. Define a connection $\nabla$ on $(\mathcal{T},g)$ by the formula

$$\nabla_U Q = (\tilde{\nabla}_U Q)_{\perp} - (\tilde{\nabla}_Q U_{\|})_{\perp} + \tfrac{1}{2}\lambda(U)Q, \quad \forall U \in \mathcal{X} \text{ and } Q \in \Gamma\mathcal{T}. \tag{63}$$

We can prove the following facts:

(i) $\nabla$ is $g$-Riemannian,
(ii) $N = \frac{1}{2}(\lambda\wedge g)$,
(iii) $K_{\|} = 0$ where $K_{\|}(U,V;..) = K(U_{\|},V_{\|};..)$, i.e., $\nabla$ is Flat in longitudinal directions,
(iv) $\forall X \in \Gamma\mathcal{D}$

$$\nabla_X Q = (L_X Q)_{\perp} + \tfrac{1}{2}\lambda(X)Q. \tag{64}$$

Now we deduce, by computation, that (ii) implies (62), so that $\nabla$ solves our problem. Conversely, (62) implies that the torsion $N$ must be given by (ii), and so $\nabla$ is unique. □

**Definition 4.5.** We say that the conformal foliation $(M,\mathcal{D},\mathcal{T},g)$ is *Conformally Flat* if, for each $m \in M$ there exists a neighborhood $\mathcal{U}$ of $m$ and a function $\Omega \in C^{\infty}(\mathcal{U})$ such that $\overline{K} = 0$, where $\overline{K}$ is the curvature associated to $\bar{g} = e^{2\Omega}g$ (and to the connection given by Theorem 4.4).

**Theorem 4.6.** *If* codim $\mathcal{D} = n+1 \leq 2$, *then* $\mathcal{T}$ *is always conformally flat. For* $n+1 \geq 4$, $\mathcal{T}$ *is conformally flat iff* $C = 0$.

**Note.** The case $n+1 = 3$ requires a special treatment (see [10, Theorem 4.3]).

Now we introduce some more definitions and computations, which we will use later in the quantization program. First note that $K_{\|} = 0$ implies that we can choose an orthonormal longitudinal parallel transversal frame (in short, an OLPT frame) $E_a$, i.e., $g(E_a, E_b) = 0$, for $a \neq b$ ; $g(E_a, E_a) = \epsilon_a (= \pm 1)$ and $\nabla_X E_a = 0, \forall X \in \Gamma\mathcal{D}$.

In the following, $E_a$ always designate such an OLPT frame for $\Gamma\mathcal{T}$, and $\nabla$ the connection given by (63). If $X_\alpha$ is a local frame for $\Gamma\mathcal{D}$ then (64) gives

$$0 = \nabla_{X_\alpha} E_a = (L_\alpha E_a)_\perp + \tfrac{1}{2}\lambda(X_\alpha)E_a$$

and, by (26), with $\lambda = -\Theta$

$$L_\alpha E_a = \tfrac{1}{2} C_\alpha E_a + \omega_\alpha^\beta(E_a)X_\beta. \tag{65}$$

**Definition 4.7.** We define the *Ricci Tensor* **Ric** $\in \mathcal{D}^{1,1}$ by the formula

$$\mathbf{Ric}(X; Q) = (c\boldsymbol{K})(X; Q) = \sum_a \epsilon_a \boldsymbol{K}(X, E_a; Q, E_a). \tag{66}$$

**Lemma 4.8.** $\mathbf{Ric}(X; Q) = \frac{1}{2}n\, \mathrm{d}\Theta(X, Q)$.

*Proof.* We make use of the following identity from [10]:

$$\operatorname{cyclsum} \boldsymbol{K}(X, Y; Z_\perp, .) = -\tfrac{1}{2} \operatorname{cyclsum} \mathrm{d}\Theta(X, Y) g(Z_\perp, .)$$

to compute $\boldsymbol{K}(X, E_a; Q, E_a)$. We have that

$$\boldsymbol{K}(X, E_a; Q, E_a) = -\tfrac{1}{2}\mathrm{d}\Theta(X, E_a)g(Q, E_a) - \tfrac{1}{2}\epsilon_a \mathrm{d}\Theta(Q, X).$$

Multiplying by $\epsilon_a$ and summing over $a = 1, \ldots, n$, we get easily the result, using $\sum_a \epsilon_a g(Q, E_a)E_a = Q$. □

**Definition 4.9.** For $Q \in \Gamma\mathcal{T}$ we define the *Transversal Divergence* of $Q$, as the function

$$\mathbf{div}_\perp Q = \sum_a \epsilon_a g(\nabla_{E_a} Q, E_a). \tag{67}$$

**Lemma 4.10.**

$$X_\alpha \cdot \mathbf{div}_\perp U_\perp = C_\alpha\, \mathbf{div}_\perp U_\perp + \tfrac{1}{2}(n-1)\, \mathrm{d}\Theta(X_\alpha, U_\perp). \tag{68}$$

*Proof.* Omiting the symbols $\sum_a$ and $\perp$ in $U_\perp$ and $\mathbf{div}_\perp$, we have

$$\begin{aligned} X_\alpha \cdot \mathbf{div}\, U &= X_\alpha \cdot \epsilon_a g(\nabla_{E_a} U, E_a) \\ &= \epsilon_a g(\nabla_{X_\alpha}\nabla_{E_a} U, E_a) + \epsilon_a g(\nabla_{E_a} U, \nabla_{X_\alpha} E_a) \\ &= \epsilon_a g(\nabla_{E_a}\nabla_{X_\alpha} U + \nabla_{[X_\alpha, E_a]} U + \boldsymbol{R}(X_\alpha, E_a)U, E_a) \\ &= \epsilon_a g(\nabla_{E_a}(\tfrac{1}{2}C_\alpha U), E_a) + \epsilon_a g(\nabla_{1/2\, C_\alpha E_a + \omega_\alpha^\beta(E_a)X_\beta} U, E_a) \\ &\quad + \mathbf{Ric}(X_\alpha; U), \end{aligned}$$

where we have used (65) for $L_\alpha E_a = [X_\alpha, E_a]$ and the fact that $\nabla_{X_\alpha} U = \frac{1}{2} C_\alpha U$ (use (64) together with (24)). So, computing we obtain

$$\begin{aligned}X_\alpha \cdot \mathbf{div}\, U &= \tfrac{1}{2}\, \epsilon_a g((E_a \cdot C_\alpha)U + C_\alpha \nabla_{E_a} U, E_a) + \tfrac{1}{2}\epsilon_a g(C_\alpha \nabla_{E_a} U, E_a)\\ &\quad + \epsilon_a \omega_\alpha^\beta(E_a) g(\nabla_{X_\beta} U, E_a) + \mathbf{Ric}(X_\alpha, U)\\ &= \tfrac{1}{2}\epsilon_a\, \mathrm{d}C_\alpha(E_a) g(U, E_a) + \tfrac{1}{2}\epsilon_a C_\alpha g(\nabla_{E_a} U, E_a) + \tfrac{1}{2}\epsilon_a C_\alpha g(\nabla_{E_a} U, E_a)\\ &\quad + \epsilon_a \omega_\alpha^\beta(E_a) g(\tfrac{1}{2} C_\beta U, E_a) + \mathbf{Ric}(X_\alpha; U)\\ &= \tfrac{1}{2}\, \mathrm{d}C_\alpha(U) + \tfrac{1}{2} C_\beta \omega_\alpha^\beta(U) + C_\alpha\, \mathbf{div}\, U + \mathbf{Ric}(X_\alpha; U)\\ &= \tfrac{1}{2}\, \mathrm{d}\Theta(U, X_\alpha) + C_\alpha\, \mathbf{div}\, U + \mathbf{Ric}(X_\alpha; U),\end{aligned}$$

where we have used the following:

$$\begin{aligned}U \cdot C_\alpha &= U(\Theta(X_\alpha))\\ &= (L_U \Theta)(X_\alpha) + \Theta([U, X_\alpha])\\ &= (i_U\, \mathrm{d}\Theta + \mathrm{d}i_U \Theta)(X_\alpha) + \Theta(-C_\alpha U - C_\alpha^\beta X_\beta)\\ &= \mathrm{d}\Theta(U, X_\alpha) - C_\beta C_\alpha^\beta \end{aligned} \tag{69}$$

with $C_\alpha^\beta = \omega_\alpha^\beta(U)$.

Finally, using Lemma 4.8, we obtain

$$\begin{aligned}X_\alpha \cdot \mathbf{div}\, U &= C_\alpha\, \mathbf{div}\, U + \tfrac{1}{2}\, \mathrm{d}\Theta(U, X_\alpha) + \tfrac{1}{2} n\, \mathrm{d}\Theta(X_\alpha, U)\\ &= C_\alpha\, \mathbf{div}\, U + \tfrac{1}{2}\,(n-1)\, \mathrm{d}\Theta(X_\alpha, U).\end{aligned}$$ □

Finally, we want to define a kind of scalar curvature for the connection $\nabla$, given by (63).

**Definition 4.11.** We define the *Scalar Curvature* of $\nabla$ by

$$\boldsymbol{S} = \boldsymbol{c} \circ \boldsymbol{c}\, \boldsymbol{K} = \boldsymbol{c}\, \mathbf{Ric}.$$

We need the longitudinal derivative of $\boldsymbol{S}$, $X_\alpha \cdot \boldsymbol{S}$. For this, we first recall the Bianchi Identity for the connection $\nabla$ on $(\mathcal{T}, g)$ (see [13, p.89])

$$\mathrm{cycl}_{(U,V,W)}\, \nabla_U \boldsymbol{R}(V, W) = \mathrm{cycl}_{(U,V,W)}\, \boldsymbol{R}([U, V], W). \tag{70}$$

Recall that $(\nabla_U \boldsymbol{R}(V, W))(Q) = \nabla_U(\boldsymbol{R}(V, W)Q) - \boldsymbol{R}(V, W)(\nabla_U Q)$. Taking the inner product with another $S \in \Gamma\mathcal{T}$, and using the properties of $\nabla$, we easily see that we can write Bianchi's Identity in the form

$$\begin{aligned}&\mathrm{cycl}_{(U,V,W)}(U \cdot \boldsymbol{K}(V, W; Q, S) - \boldsymbol{K}(V, W; Q, \nabla_U S) - \boldsymbol{K}(V, W; \nabla_U Q, S))\\ &\quad = \mathrm{cycl}_{(U,V,W)}\, (\boldsymbol{K}([U, V], W; Q, S)).\end{aligned} \tag{71}$$

Now put $U = X_\alpha$, $V = E_c$, $W = E_d$, $Q = E_a$, $S = E_b$ in (71) and compute the contractions in the pairs of indices $(d, b)$ and $(c, a)$, respectively, using formula (69) and the fact that $\boldsymbol{N}_\perp = 0$ (see Definition 4.2(ii)). After a tedious calculation we conclude with the following lemma.

**Lemma 4.12.**

$$X_\alpha S = C_\alpha S - n(2V_\alpha^\beta \cdot C_\beta + C_\beta\, \mathbf{div}_\perp V_\alpha^\beta + C_\beta \omega_\gamma^\beta (V_\alpha^\gamma) + \Delta C_\alpha), \tag{72}$$

*where*

$$V_\alpha^\beta = \sum_a \epsilon_a \omega_\alpha^\beta (E_a) E_a = \eta^{ac} \omega_\alpha^\beta (E_c) E_a \tag{73}$$

*is the transversal vector field metric-equivalent to the* 1*-form* $\omega_\alpha^\beta$, *and* $\Delta C_\alpha$ *is the* Transversal Laplacian *of* $C_\alpha$, *defined in the following definition.*

**Definition 4.13.** For a function $\phi \in C^\infty(M)$ we define its *Transversal Laplacian* $\Delta$ by

$$\Delta\phi = \mathbf{div}_\perp\, \mathbf{grad}_\perp \phi \tag{74}$$

where $\mathbf{grad}_\perp \phi = \sum_a \epsilon_a\, \mathrm{d}\phi(E_a) E_a = \eta^{ab}(E_a \cdot \phi) E_b$ is the *Transversal Gradient* of $\phi$.

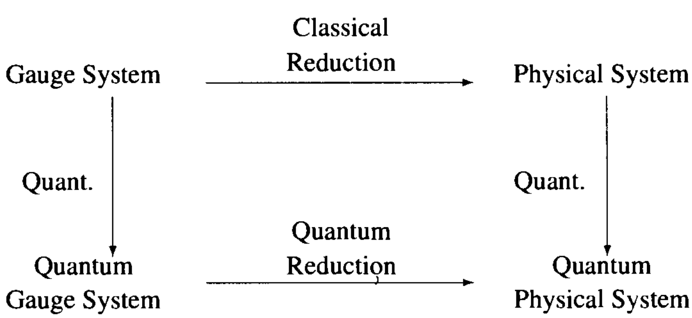
We can compute that

$$\Delta\phi = \sum_a \epsilon_a g(\nabla_{E_a} \mathbf{grad}_\perp \phi, E_a) \tag{75}$$

$$= \eta^{ab}(E_a E_b - \nabla_{E_a} E_b)\phi. \tag{76}$$

## 5. Quantization

When we face the problem of quantizing a gauge system, two different approaches are conceivable, in principle. We can reduce the gauge system to the physical system (if possible) and quantize or, either, quantize the gauge system directly and then reduce by some *Quantum Reduction Process*. Hopefully, these two processes must be consistent, i.e., schematically the following diagram must commute: (see discussion in [8,5])




As we have seen in the preceding sections, the reduced physical system (when exists) is characterized by a conformal class of physical superhamiltonians,

$$h = \mathrm{e}^w (g + J_u + v), \quad w \in C^\infty(m)$$

Moreover, the dynamical constraint remains after reduction: $h = 0$. So, the proper way of quantizing this relativistic physical system is through a *Conformal Klein–Gordon* type equation.

**Note**. Recall that an equation of type $\mathcal{D}_g \Phi = 0$, for a field $\Phi$ (where $\mathcal{D}_g$ is an operator constructed from the metric $g$), is called *Conformally Invariant* if there exists a number $k \in \mathbb{R}$ (called the *Conformal Weight of the Equation or of the Field*) such that $\Phi$ is a solution of $\mathcal{D}_g \Phi = 0$ if and only if $\overline{\Phi} = e^{k\Omega}\Phi$ is a solution of $\mathcal{D}_{\bar{g}}\overline{\Phi} = 0$, where $\mathcal{D}_{\bar{g}}$ is the same operator but now constructed from the metric $\bar{g} = e^{\Omega} g$, (see [17]).

Now we define the *Quantum Operators* corresponding, respectively to $g$, $J_u$ and $v$, by

$$\hat{g} = -\Delta_c = -\Delta_g - \xi S_g, \tag{77}$$

$$\hat{J}_u = (1/i)\,(u + \tfrac{1}{2}\,\mathbf{div}_g\, u), \tag{78}$$

$$\hat{v} = v \tag{79}$$

acting on $C^\infty(m)$, where $\Delta_c \equiv \Delta_g + \xi S_g$ is the *Conformal Laplacian* of the metric $g$ (here, $\Delta_g$ and $S_g$ are, respectively, the usual Laplacian and scalar curvature of $g$), and $\xi = (n-1)/4n$.

Then it is easy to see that the *Conformal Klein–Gordon Equation*

$$\hat{h}\psi = (\hat{g} + \hat{J}_u + \hat{v})\psi = 0 \tag{80}$$

is conformally invariant with weight $k = (n-1)/4$.

Of course, now we must face the problem of the possibility of construction of an Hilbert space from solutions of (80), as well as the interpretation of the theory (one-particle, second quantization, etc.) (see [7,2], for which we defer the discussion of these subjects).

What about *Constraint Quantization*? Here we adopt Kuchař's philosophy, reinforced by the essential unique character of the objects defined in Section 4, namely, the connection $\nabla$, the corresponding scalar curvature $S$ (Definition 4.11) and transversal Laplacian $\mathbf{\Delta}$ (Definition 4.13).

So, we quantize the supermomentum constraints $H_\alpha$ by the following operators:

$$\hat{H}_\alpha = (1/i)\,(X_\alpha - kC_\alpha) \tag{81}$$

and the transversal superhamiltonian $H_\perp$, by

$$\hat{H}_\perp = \hat{g} + \hat{J}_{U_\perp} + \hat{V} \tag{82}$$

with

$$\hat{g} = -\mathbf{\Delta} - \xi S, \tag{83}$$

$$\hat{J}_{U_\perp} = (1/i)\,(U_\perp + \tfrac{1}{2}\,\mathbf{div}_\perp U_\perp), \tag{84}$$

$$\hat{V} = V, \tag{85}$$

acting on $C^\infty(M)$.

Then the classical constraints are imposed at quantum level as the *Quantum Constraints*

$$\hat{\boldsymbol{H}}_\alpha \psi = 0 = \hat{\boldsymbol{H}}_\perp \psi, \quad \psi \in C^\infty(M). \tag{86}$$

As we have said, this *Quantum Reduction Process* must be consistent with the first approach to quantization (see discussion in [8,5]). This is implied by the following lemmas and theorems.

**Theorem 5.1.**

$$(1/\mathrm{i})\,[\hat{\boldsymbol{H}}_\alpha, \hat{\boldsymbol{H}}_\beta] = \Gamma^\gamma_{\alpha\beta} \hat{\boldsymbol{H}}_\gamma,$$

*where* $\Gamma^\gamma_{\alpha\beta} \in C^\infty(M)$ *are given by* (9).

*Proof.* Compute, using definitions and (37). □

**Lemma 5.2.**

$$(1/\mathrm{i})\,[\hat{\boldsymbol{J}}_{U_\perp}, \hat{\boldsymbol{H}}_\alpha] = C_\alpha \hat{\boldsymbol{J}}_{U_\perp} + \omega^\beta_\alpha(U_\perp) \hat{\boldsymbol{H}}_\beta,$$

*where* $C_\alpha = \Theta(X_\alpha)$ *and* $\omega^\beta_\alpha(U_\perp)$ *is given by* (30).

*Proof.* Using definition (84) we compute the commutator in the LHS of the above equation, and conclude that

$$(1/\mathrm{i})\,[\hat{\boldsymbol{J}}_{U_\perp}, \hat{\boldsymbol{H}}_\alpha]\phi = (C_\alpha \hat{\boldsymbol{J}}_{U_\perp} + \omega^\beta_\alpha(U_\perp)\hat{\boldsymbol{H}}_\beta)\psi + (k/\mathrm{i})\,(U_\perp \cdot C_\alpha + \omega^\beta_\alpha(U_\perp) C_\beta)$$
$$+ \tfrac{1}{2}\,(X_\alpha \cdot \mathbf{div}_\perp U_\perp - C_\alpha\, \mathbf{div}_\perp U_\perp)\psi.$$

However, by (69), $U_\perp \cdot C_\alpha + C_\beta \omega^\beta_\alpha(U_\perp) = \mathrm{d}\Theta(U_\perp, X_\alpha)$, while Lemma 4.10 gives $X_\alpha \cdot \mathbf{div}_\perp U_\perp = C_\alpha\, \mathbf{div}_\perp U_\perp + \tfrac{1}{2}(n-1)\,\mathrm{d}\Theta(X_\alpha, U_\perp)$, and so we see that the last sum in the RHS is zero. □

**Lemma 5.3.**

$$(1/\mathrm{i})\,[\hat{\boldsymbol{g}}, \hat{\boldsymbol{H}}_\alpha] = C_\alpha \hat{\boldsymbol{g}} + 2(\hat{\boldsymbol{V}}^\beta_\alpha - \tfrac{1}{2}\mathrm{i}\, \omega^\beta_\gamma(V^\gamma_\alpha))\hat{\boldsymbol{H}}_\beta,$$

*where* $V^\beta_\alpha$ *is defined in* (73), *and, as usual* (*see* [1])

$$\hat{\boldsymbol{V}}^\beta_\alpha = (1/\mathrm{i})\,(V^\beta_\alpha + \tfrac{1}{2}\, \mathbf{div}_\perp V^\beta_\alpha) \tag{87}$$

*with* $\mathbf{div}_\perp V^\beta_\alpha$ *as in Definition* 4.9, *Eq.* (67).

*Proof.* First we compute that

$$(1/\mathrm{i})\,[\hat{\boldsymbol{g}}, \hat{\boldsymbol{H}}_\alpha] = [\boldsymbol{\Delta}, X_\alpha] - k[\boldsymbol{\Delta}, C_\alpha] + \xi[\boldsymbol{S}, X_\alpha] \tag{88}$$

and it is easy to see that

$$[\boldsymbol{\Delta}, C_\alpha] = 2\mathrm{i}\, \widehat{\mathbf{grad}}_\perp C_\alpha = 2(\mathbf{grad}_\perp C_\alpha + \tfrac{1}{2}\, \Delta C_\alpha) \tag{89}$$

and

$$[S, X_\alpha] = -X_\alpha \cdot S. \tag{90}$$

So we must compute $[\boldsymbol{\Delta}, X_\alpha]$. For this, we use Eqs. (75), (76) for $\boldsymbol{\Delta}\phi$ and compute that

$$[\boldsymbol{\Delta}, X_\alpha]\phi = \eta^{ab}(E_a E_b X_\alpha - X_\alpha E_a E_b)\phi + \eta^{ab}(X_\alpha \nabla_{E_a} E_b - \nabla_{E_a} E_b X_\alpha)\phi. \tag{91}$$

Now we commute the derivatives and use systematically (65), to get

$$\begin{aligned} X_\alpha E_a E_b \phi = {} & E_a E_b X_\alpha \phi + C_\alpha E_a E_b \phi + \tfrac{1}{2}(E_a \cdot C_\alpha) E_b \phi + \omega_\alpha^\beta(E_b) E_a X_\beta \phi \\ & + (E_a \cdot \omega_\alpha^\beta(E_b)) X_\beta \phi + \omega_\alpha^\beta(E_a) E_b X_\beta \phi \\ & + \tfrac{1}{2}\, C_\beta \omega_\alpha^\beta(E_\alpha) E_b \phi + \omega_\alpha^\beta(E_a) \omega_\beta^\gamma(E_b) X_\gamma \phi \end{aligned}$$

and

$$(X_\alpha \nabla_{E_a} E_b - \nabla_{E_a} E_b X_\alpha)\phi = (C_\alpha \nabla_{E_a} E_b + \boldsymbol{R}(X_\alpha, E_a) E_b + \omega_\alpha^\beta(\nabla_{E_a} E_b) X_\beta)\phi.$$

Substituting these last two identities in (91), we obtain

$$\begin{aligned} [\boldsymbol{\Delta}, X_\alpha]\phi = {} & -C_\alpha\, \boldsymbol{\Delta}\phi - (2V_\alpha^\beta + \mathbf{div}_\perp V_\alpha^\beta + \omega_\gamma^\beta(V_\alpha^\gamma)) X_\beta \phi \\ & -\tfrac{1}{2}(\mathbf{grad}_\perp C_\alpha + C_\beta V_\alpha^\beta - 2\eta^{ab} \boldsymbol{R}(X_\alpha, E_a) E_b). \end{aligned}$$

Now we use the fact

$$\begin{aligned} \eta^{ab} \boldsymbol{R}(X_\alpha, E_a) E_b & = \eta^{ab} \sum_c \epsilon_c \boldsymbol{K}(X_\alpha, E_a; E_b, E_c) E_c \\ & = -\sum_c \epsilon_c \eta^{ab} \boldsymbol{K}(X_\alpha, E_a; E_b, E_c) E_c \\ & = \tfrac{1}{2}\, n \textstyle\sum_c \epsilon_c (E_c \cdot C_\alpha + C_\beta \omega_\alpha^\beta(E_c)) E_c \\ & = \tfrac{1}{2}\, n\, (\mathbf{grad}_\perp C_\alpha + C_\beta V_\alpha^\beta), \end{aligned}$$

where we have used (69), Lemma 4.10 and (73).

By [13, p. 151], we have

$$\mathbf{div}_\perp V_\alpha^\beta = \mathbf{div}_\perp \omega_\alpha^\beta = \eta^{ab}(\nabla_{E_a} \omega_\alpha^\beta)(E_b) = \eta^{ab}(E_a \cdot \omega_\alpha^\beta(E_b) - \omega_\alpha^\beta(\nabla_{E_a} E_b))$$

and so, substituting and calculating, we have

$$\begin{aligned} [\boldsymbol{\Delta}, X_\alpha]\phi = {} & -C_\alpha \boldsymbol{\Delta}\phi - C_\alpha \xi S \phi + 2(\hat{\boldsymbol{V}}_\alpha^\beta - \tfrac{1}{2}\mathrm{i}\, \omega_\gamma^\beta(V_\alpha^\gamma)) \hat{\boldsymbol{H}}_\beta \phi \\ & -k(2V_\alpha^\beta \cdot C_\beta + C_\beta\, \mathbf{div}_\perp V_\alpha^\beta + C_\beta \omega_\gamma^\beta(V_\alpha^\gamma))\phi \\ & +2k\, \mathbf{grad}_\perp C_\alpha \phi + C_\alpha \xi S \phi. \end{aligned}$$

By (89)–(91), we have

$$\begin{aligned} (1/\mathrm{i})\, [\hat{\boldsymbol{g}}, \hat{\boldsymbol{H}}_\alpha]\phi & = C_\alpha \hat{\boldsymbol{g}} \phi + 2(\hat{\boldsymbol{V}}_\alpha^\beta - \tfrac{1}{2}\mathrm{i}\, \omega_\gamma^\beta(V_\alpha^\gamma)) \hat{\boldsymbol{H}}_\beta \phi \\ & = -k(2V_\alpha^\beta \cdot C_\beta + C_\beta\, \mathbf{div}_\perp V_\alpha^\beta + C_\beta \omega_\gamma^\beta(V_\alpha^\gamma))\phi \\ & \quad + C_\alpha \xi S \phi - k \boldsymbol{\Delta} C_\alpha - \xi (X_\alpha \cdot S)\phi \end{aligned}$$

and so, it suffices to prove that

$$\xi X_\alpha \cdot S = \xi C_\alpha S - k(2V_\alpha^\beta \cdot C_\beta + C_\beta \,\mathbf{div}_\perp V_\alpha^\beta + C_\beta \omega_\gamma^\beta (V_\alpha^\gamma) + \mathbf{\Delta} C_\alpha)$$

which is precisely Lemma 4.12. □

Now, collecting the above two lemmas, together with (13), we finally have the following theorem.

**Theorem 5.4.**

$$(1/\mathrm{i})\,[\hat{\boldsymbol{H}}_\perp, \hat{\boldsymbol{H}}_\alpha] = C_\alpha \hat{\boldsymbol{H}}_\perp + \boldsymbol{C}_\alpha^\beta \hat{\boldsymbol{H}}_\beta,$$

*where* $\boldsymbol{C}_\alpha^\beta = 2\hat{\boldsymbol{V}}_\alpha^\beta - \mathrm{i}\omega_\gamma^\beta(\boldsymbol{V}_\alpha^\gamma) + \omega_\alpha^\beta(U_\perp)$.

So, Theorems 5.1 and 5.4 show that we have a closed commutator algebra of quantum operator constraints, with the structure functions appearing on the left of that quantum operators. This implies the consistency of the above quantization process. Moreover, we recover the physical theory described by the *Conformal Klein–Gordon Equation* (80). In fact, start with a wave function $\Psi \in C^\infty(M)$ that solves the quantum constraints (86). Assume also that the foliation $\mathcal{F}$ is simple, and that $\Theta$ is $\mathrm{d}_{\mathcal{F}} - exact$, i.e., $\exists \Omega \in C^\infty(M)$ such that $\Theta = -\mathrm{d}_{\mathcal{F}}\Omega$. Then we easily prove that the rescaled wave function $\overline{\Psi} = \mathrm{e}^{k\Omega}\Psi$ is a basic function and so, descends to a wave function $\overline{\psi} \in C^\infty(m)$. Now, as in [6], we can prove that the transversal superhamiltonian $\hat{\boldsymbol{H}}_\perp$ has conformal weight 1, when acting on wave functions of conformal weight $k$, i.e.,

$$\hat{\overline{\boldsymbol{H}}}_\perp \overline{\Psi} = \mathrm{e}^{(k+1)\Omega} \hat{\boldsymbol{H}}_\perp \Psi, \tag{92}$$

where $\bar{\Psi} = \mathrm{e}^{k\Omega}\Psi$, and $\hat{\overline{H}}_\perp$ is the transversal superhamiltonian constructed from the rescaled metric $\bar{g} = \mathrm{e}^{\Omega} g$. So we see that (86) $\Rightarrow \hat{\boldsymbol{H}}_\perp \Psi = 0 \Rightarrow \hat{\overline{H}}_\perp \overline{\Psi} = 0$, and, since $\overline{\Psi}$ is basic, this equation implies that

$$\hat{\overline{\boldsymbol{h}}}\, \overline{\psi} = 0, \tag{93}$$

where $\overline{\psi}$ is the induced wave function on $m$. In this way, we recover the physical conformal Klein–Gordon equation (80).

## Appendix A

Hereafter we adopt the following notations, related to decompositions (21)–(24):

$Q^A$ are local coordinates on $M$.
$\partial_A = \partial/\partial Q^A$, and $\partial^A = \mathrm{d}Q^A$, $A = 0, \ldots, N$.
$X_\alpha = X_\alpha^A \partial_A$, a local frame field for $\Gamma\mathcal{D}$, $\alpha = 1, \ldots, v$.
$Q_a = Q_a^A \partial_A$, a local frame field for $\Gamma\mathcal{T}$, $a = 0, \ldots, n$.
$(\theta^\alpha, \theta^a)$, a dual coframe for $\mathcal{D} \oplus \mathcal{T}$, so that $\theta^\alpha \in \Gamma\mathcal{D}^*$, $\theta^a \in \Gamma\mathcal{T}^* \cong \Gamma\mathcal{D}^0$ (we may assume that $\theta^a$ are closed 1-forms), $\theta^\alpha(X_\beta) = \delta_\beta^\alpha$, $\theta^a(Q_b) = \delta_b^a$, $\theta^\alpha(Q_a) = 0 = \theta^a(X_\alpha)$.

$G = G^{AB}\partial_A \otimes \partial_B$, a contravariant metric on $M$.
$\tilde{g} = G|(\mathcal{D}^0 \cong \mathcal{T}^*)$.

The projector $P_\perp : TM \to \mathcal{T}$ can be written in the form

$$P_\perp = Q_a \otimes \theta^a(\cdot) = Q_a^B Q_A^a \partial_B \otimes \partial^A = \mathcal{P}_A^B \partial_B \otimes \partial^A$$

and the transversal metric $G_\perp$ is

$$\begin{aligned} G_\perp &= P_\perp \otimes P_\perp(G) = Q_a \otimes \theta^a \otimes Q_b \otimes \theta^b (G^{AB}\partial_A \otimes \partial_B) \\ &= G^{AB} Q_a^C Q_A^a Q_b^E Q_B^b \partial_C \otimes \partial_E = G^{AB} \mathcal{P}_A^C \mathcal{P}_B^E \partial_C \otimes \partial_E. \end{aligned}$$

From

$$\tilde{g}^{ab} = G(\theta^a, \theta^b) = G^{AB}\partial_A \otimes \partial_B(Q_C^a \partial^C, Q_D^b \partial^D) = G^{AB} Q_A^a Q_B^b,$$

we can also write

$$G_\perp = \tilde{g}^{ab} Q_a^C Q_b^E \partial_C \otimes \partial_E. \tag{94}$$

Now it is easy to see that

$$L_\alpha \theta^a \in \mathcal{D}^0 \cong \mathcal{T}^*, \quad \forall \alpha, a. \tag{95}$$

We compute now $L_\alpha P_\perp = L_\alpha(Q_a \otimes \theta^a)$: first we decompose $L_\alpha Q_a = t_{\alpha a}^b Q_b + l_{\alpha a}^\beta X_\beta$. However, we have

$$0 = L_\alpha \langle \theta^b, Q_a \rangle = \langle L_\alpha \theta^b, Q_a \rangle + \langle \theta^b, L_\alpha Q_a \rangle$$

and by (94), we can put $L_\alpha \theta^b = r_{\alpha d}^b \theta^d$ and deduce, from the last equation, that $t_{\alpha a}^b = -r_{\alpha a}^b$. So, reuniting these information, we easily compute that

$$L_\alpha P_\perp = l_{\alpha a}^\beta X_\beta \otimes \theta^a, \tag{96}$$

where $l_{\alpha a}^\beta = \theta^\beta(L_\alpha Q_a)$.

Now we compute $L_\alpha U_\perp$, with $U_\perp = P_\perp(U)$ and $U \in \mathcal{X}(M)$. By (28)

$$\begin{aligned} L_\alpha U_\perp &= L_\alpha(P_\perp \bullet U) \\ &= (L_\alpha P_\perp) \bullet U + P_\perp \bullet (L_\alpha U) \\ &= (l_{\alpha a}^\beta X_\beta \otimes \theta^a)(u^a Q_a + u^\beta X_\beta) + P_\perp(C_\alpha + \text{long. vector}) \quad \text{by (12)} \\ &= l_{\alpha a}^\beta u^a X_\beta + C_\alpha U_\perp, \end{aligned}$$

and recalling the definition of the 1-form $\omega_\alpha^\beta$ in (29), it is easy to see that $\omega_\alpha^\beta = l_{\alpha a}^\beta \theta^a$ and so we can write $L_\alpha U_\perp$ also in the form (30).

Now we compute $L_\alpha G_\perp$. Again by (28), we have, with $\mathcal{P}_\perp = P_\perp \otimes P_\perp$

$$\begin{aligned} L_\alpha G_\perp &= L_\alpha(\mathcal{P}_\perp \bullet G) \\ &= \mathcal{P}_\perp \bullet (L_\alpha G) + (L_\alpha \mathcal{P}_\perp) \bullet G \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}_\perp \bullet (C_\alpha G + \text{terms in } \boldsymbol{I}) + L_\alpha (P_\perp \otimes P_\perp) \bullet G \\
&= C_\alpha G_\perp + (L_\alpha P_\perp \otimes P_\perp + P_\perp \otimes L_\alpha P_\perp) \bullet G \\
&= C_\alpha G_\perp + l^\beta_{\alpha a} X_\beta \otimes \theta^a \otimes Q_b \otimes \theta^b + Q_b \otimes \theta^b \\
&\quad \otimes l^\beta_{\alpha a} X_\beta \otimes \theta^a (\tilde{g}^{cd} Q_c \otimes Q_d) \\
&= C_\alpha G_\perp + l^\beta_{\alpha a} \tilde{g}^{ab} X_\beta \otimes Q_b + l^\beta_{\alpha a} \tilde{g}^{ba} Q_b \otimes X_\beta ,
\end{aligned}
$$

where we have used (11) and the fact that $\theta^a \in \mathcal{D}^0$. Now, let $V^\beta_\alpha = G(\omega^\beta_\alpha, .) \in \Gamma\mathcal{T}$ the transversal vector field, metric-equivalent to the transversal 1-form $\omega^\beta_\alpha$, given by (29). We compute that $V^\beta_\alpha = l^\beta_{\alpha a} \tilde{g}^{ab} Q_b$, and so we see that we can write $L_\alpha G_\perp$ in the form (31).

As in Section 3, we assume that $\tilde{g} = G|\mathcal{D}^0 = G_\perp|\mathcal{D}^0$ is nondegenerate, and let $g$ be the corresponding transversal covariant metric on $\mathcal{T}$. We want to compute $L_\alpha g$. For this, we first note that $L_\alpha g$ must be a covariant transversal vector field, since $g = g_{ab}\theta^a \otimes \theta^b$ and $L_\alpha \theta^a \in \mathcal{D}^0$. So, we compute $(L_\alpha \tilde{g})|\mathcal{T}$, since it suffices to compute $L_\alpha g$. We have

$$
\begin{aligned}
(L_\alpha \tilde{g})(\theta^a, \theta^b) &= (L_\alpha G_\perp)(\theta^a, \theta^b) \\
&= (C_\alpha G_\perp + V^\beta_\alpha \vee X_\beta)(\theta^a, \theta^b) \\
&= C_\alpha G_\perp(\theta^a, \theta^b) \quad \text{since } \theta^a \in \mathcal{D}^0 \\
&= C_\alpha \tilde{g}(\theta^a, \theta^b) \quad \text{by (94)}
\end{aligned}
$$

and so, defining $L_\alpha g$ as

$$
(L_\alpha g)(Q_a, Q_b) = g_{ac} g_{bd} (L_\alpha \tilde{g})(\theta^a, \theta^b) \tag{97}
$$

we conclude that $L_\alpha g = -C_\alpha g$. Notice that if $G$ is a covariant metric on $M$, such that $G(\mathcal{D}, \mathcal{T}) = 0$, then definition (97) of $L_\alpha g$ is equivalent to

$$
(L_\alpha g)(Q_a, Q_b) = (L_\alpha G)(Q_a, Q_b). \tag{98}
$$

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